2017 Oral Exam: Probability and Statistics
Individual

Problem 1. *Let X be a random variable with finite variance. Denote by m, μ, σ the median, mean and standard deviation of X:*

$$m := \inf\{c : \mathbb{P}[X \leq c] \geq 1/2\}, \quad \mu = \mathbb{E}[X], \quad \sigma^2 = \mathbb{E}[(X-\mu)^2].$$

Show that $|m - \mu| \leq \sigma$.

Problem 2. *Let $(X_n)_{n\geq 1}$ be a sequence of non-negative random variables. Let $(\mathcal{F}_n)_{n\geq 1}$ be a filtration (i.e. a sequence of increasing σ-algebras). Assume that*

$$\mathbb{E}[X_n \,|\, \mathcal{F}_n] \to 0, \quad \textit{in probability.}$$

Show that

$$X_n \to 0, \quad \textit{in probability.}$$

Is it true reversely? If yes, prove it; if not, give a counterexample.

Problem 3. *Let $X_1, ..., X_n$ be independent random variables following common Poisson distribution with mean λ. Let $\eta = e^{-\lambda}$. Does there exist a uniformly unbiased minimum variance estimator UMVUE of η? (Recall that an estimator is UMVUE if it is unbaised estimator and has smallest variance among all unbiased estimators.) If yes, find it; if no, prove it.*